A Construction of the Regular Polytopes of All Dimensions

Thomas Eliot*

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Abstract

The Platonic Solids are the most symmetrical possible objects. They are the dice of the gods, these beautiful shapes, and each dimension has its own set. We shall construct them fully in every dimension by means of examining their symmetries.

1 What is a Platonic Solid?

A *polytope* is a shape with flat sides. It is *convex* if, given any two points a and b in the object, the line segment connecting them, $ta + (1-t)b$ where t ranges from 0 to 1, is also in the object. A *symmetry*, or *isometry*, is a transformation of some shape that preserves distance-as a direct result of this, it also preserves angles. Put in plain terms, an object looks the same after you perform an isometry on it. In 3 dimensions, the possible symmetries are reflections, rotations, translations, and screw translations (a translation and a rotation).

Polytopes of dimension n have sides-bounding hyperplanes-of dimension $n-1$ which are themselves polytopes. They in turn have sides of dimension $n-2$, and so on, down to dimension 1 edges bounded by dimension 0 vertices. If you select one of each of these sides, each bounding the dimension one higher than it, this selection is called a $flag$. If any flag can be sent to any other flag by some isometry of the whole shape, then this polytope is *regular*; we say that its symmetries act *transitively* on its flags.

The *Platonic Solids* are the convex regular polytopes in dimension 3.

1.1 The Platonic Solids and Other Dice: Beautiful Symmetries

The *Platonic Solids* are the 4 sided tetrahedron, the 6 sided cube, the 8 sided octahedron, the 12 sided dodecahedron, and the 20 sided icosahedron. They are named for Plato, who associated each with an element: the tetrahedron fire, the cube earth, the octahedron air, the icosahedron water, while the dodecahedron was thought to describe the locations of the stars in the sky. Theaetetus may have been the first to prove their uniqueness, but it is certainly Euclid who wrote the most thorough description and construction of them in his *Elements*: it is his proof that we will construct here, and extend to higher dimensions.

Certain readers may recognize these as dice they have used, and may notice that one seems to be missing-the 10 sided deltohedron. If you observe it closely though, you will see that two of the edges on any given face are shorter than the other two edges, and thus any flag containing one cannot be sent to the other by any isometry.

INSERT IMAGES HERE, FIRST OF THE PLATONIC SOLIDS, THEN ONE OF THE D10

Since the isometries of the deltohedron only act transitively on its 2 dimensional faces, whereas the isometries of a Platonic solid act transitively on their faces of every dimension, and so they are more symmetric. Their symmetry groups are more beautiful and complexly interrelated. The set of actions of performing a symmetry form a *group*; the group action is concatenation of symmetries.

*Willamette University. This research was done at Pennsylvania State University's Mathematics Advanced Study Semester in Fall 2009 under the supervision of Sergei Tabachnikov and Anatole Katok.

The *dual* of an n-dimensional polytope is formed by replacing every face of dimension 0 with a face of dimension $n-1$ and vice versa, faces of dimension 1 with dimension $n-2$ and vice versa, etcetera until all faces have been swapped. There is then a natural isometry between the flags for a polytope and its dual, and so the symmetry group of a polytope and its dual are the same. Thus the self-dual terahedron defines a single symmetry group S_4, the octahedron and cube define O_h, and the dodecahedron and icosahedron define I_h. Examining these groups is beyond the scope of this paper, but any algebraicist will be able to tell you hours of fascinating information about them.

1.2 A Construction of the Regular Polygons of Dimension Two

Let us consider this expanded definition in two dimensions. We start with the $2-1=1$ dimensional edges, and then select a $2-2=0$ dimensional vertex, and then we're done: we have our flag. As long as any of these can be sent to any other of these by some isomorphism, we have a regular polytope. It is immediately apparent that these are the regular polygons: any number of sides arrange regularly in a cycle creates the needed flags. If the angles between the sides are not the same, then the isometries will not act transitively, which excludes things like non-square parallelograms.

1.3 A Construction of the Regular Polyhedra in $\mathbb{R}^3$, and the accidental construction of several tilings of the plane

If we are to construct 3-dimensional regular polytopes, any 2 dimensional face of a 3 dimensional regular polytope must be a 2 dimensional regular polytope. Thus we can construct the regular polytopes in $\mathbb{R}^3$ by grabbing the regular polytopes in $\mathbb{R}^2$ and trying to fit them around a vertex: since all vertices must appear identical, if we can describe one we've described the whole shape. The formula for the internal angle of a regular polygon is $\frac{(p-2)\pi}{p}$ where p is the number of sides it has, so let's see what our possibilities are. If our 2-d polygon has three sides, we have $\pi/3$ as our angle, if four sides $\pi/2$, if five $3\pi/5$, if six $2\pi/3$ and so on. Now before we try to fit them around a vertex, we must define "fit." We say they "fit" around a vertex if the product of the internal angles times the number of polygons we're trying to fit in is less than 2π. If it equals 2π it tiles the space, if it is greater than 2π then it (possibly, but that's beyond the scope of this paper) tiles hyperbolic space.

INSERT SOME PICTURES OF THAT STUFF HERE MAYBE

So let's see what we can build. We can fit three triangles around a vertex, or four, or five. These create the tetrahedron, octahedron, and icosahedron, respectively. If we try to fit six triangles around a vertex, we get a tiling of the plane. We can fit three squares around a vertex; this is the cube. If we try to fit four, we tile the plane. We can fit three pentagons around a vertex, this is the dodecahedron. Note that we already knew the cube and dodecahedron existed because they are the duals of the octahedron and icosahedron, respectively, so it's good our system constructed them! We've now constructed all the regular polytopes in $\mathbb{R}^3$ so our system had better not construct any more. Let's see if it does.

We cannot fit four pentagons around a vertex; their angles sum to over 2π. Can we build anything using hexagons? It would have to have more than two polygons per vertex for obvious reasons, and so has at least three, but then its angles sum to exactly 2π and we have our third tiling of the plane. If we can't build anything out of hexagons we definitely can't build anything out of heptagons or higher, so we're done! Our strategy works perfectly.

1.4 Schläfli Symbols

At this point an aside is necessary to explain what is probably the most useful notation system for what we're up to. Ludwig Schläfli (1814-1895), the aptly titled "discoverer of the fourth dimension"[1] invented a notation system for regular polytopes that creates a symbol

$\{p,q\}$ for any three dimensional polytope where each face is a regular $p$gon and there are q of them around each vertex, a symbol $\{p,q,r\}$ for a four dimensional polytope where each cell is a $\{p,q\}$*tope* and there are r of them around an edge, and so on. Recall from 1.1 that the dual of a regular polytope is a regular polytope; thus we see that if $\{p,q,r,s\}$ is a regular polytope then $\{p,q,r\}\{q,r,s\}\{r,q,p\}\{s,r,q\}\{p,q\}\{q,p\}\{q,r\}\{r,q\}\{r,s\}$ and $\{s,r\}$ are all regular polytopes. This continues to hold, and expand appropriately, for the higher dimensions. This will be useful in section 4.2, when we construct the regular polytopes of dimensions 6 and higher.

2 A Construction of the Regular Polytopes in $\mathbb{R}^4$

Now that we know how to build our polytopes, let's start doing so using the ones we know in $\mathbb{R}^3$. In order to do this, we need to know the interior angle, also called the dihedral angle, of each of our regular polyhedra. The interior angle of a 3 dimensional polyhedron is the angle between two of its 2 dimensional faces, as defined in the usual way: the angle between the normal vectors of the two planes. These are calculated using the formula[2]:

$$\cos\theta = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$$

Where a, b, c are the angles of the three faces meeting at a vertex, and θ is the dihedral angle. Now, since we're looking at a regular polyhedron, $a = b = c$. Thus our formula for the dihedral angle θ of a regular polytope in $\mathbb{R}^n$ is simply

$$\cos\theta = \frac{\cos\phi - \cos\phi\cos\phi}{\sin\phi\sin\phi}$$

where ϕ is the dihedral angle of the $n-1$ dimensional polytopes that make up the faces of the n dimensional polytope.

The tetrahedron, made up of equilateral triangles with angle $\frac{\pi}{6}$ has a dihedral angle of $cos^{-1}(1/3) \approx 70.5$. We can put three of these around a vertex, totalling 211.5 degrees and corresponding to $\{3,3,3\}$ the four dimensional simplex. We can also put four around a vertex, totaling 282 degrees and corresponding to $\{3,3,4\}$ the four dimensional cross polytope, also called the cocube. We can also put five around a vertex, barely squeezing in at 352.5 degrees is $\{3,3,5\}$ the magnificent, enormous 600 cell.

The three-dimensional cube $\{4,3\}$ clearly has dihedral angle of $\pi/2$. We can fit three of these around an edge, giving us the hypercube, $\{4,3,3\}$. We cannot then fit four of them around an edge, if we try, we form a *honeycomb*: a tiling of three dimensional space. If we cannot fit four of these we definitely cannot fit four of anything bigger, and thus we're almost done. The only remaining question is what more regular 3d polytopes can we fit three of around an edge?

The dodecahedron, made up of pentagons with interior angles $\frac{3\pi}{5}$, has interior angle $\approx 116.6^o$. Thus we can fit three dodecahedrons around an edge, giving us the dual of the 600 cell, the slightly less brobdingnagian but still overwhelming 120 cell, $\{5,3,3\}$. What else can we fit?

Before we continue, an aside is necessary. You may have noticed a flaw in our earlier dihedral formula. Namely, it requires three (and exactly three) faces to meet at a vertex, but there are Platonic solids with more than three faces meeting at a vertex: the octahedron has four triangles, the icosahedron has five. Fortunately for us the icosahedron also obviously has interior angle of greater than 120^o, so the only object we need to inspect here is the octahedron. This is where a bit of clever thinking comes in. The octahedron has four faces meeting at each vertex, true. But imagine if we slice it in half, giving us a pyramid with four triangular faces and a square base. Then inspect any of the four vertices on the base: at any of those only 3 faces meet, two triangles and the square. Thus our formula is

$$\cos(\theta) = \frac{\cos(\pi/2) - \cos^2(\pi/3)}{\sin^2(\pi/3)}$$

and so $\theta \approx 109.5^o$.

The final polytope is unique to the fourth dimension, and is often considered the most beautiful because of this fact. Surely it has inspired more Pennsylvanian works of art than any other, though if we expand this to include Spanish art[3] it is matched by the poetically named *tesseract*, or 4-cube. While I personally prefer the latter's constant orthogonal angles and simple, room-like chambers, there is certainly beauty to be found in this last shape: the Octacube, or 24 cell.

The Octahedron $\{3, 4\}$ has interior angle of approximately 109.5 degrees. This means we can fit three (and only three) of them around an edge, and create a new, strange shape $\{3, 4, 3\}$ that has no equivalent in any other dimensions.

We have now run out of $3-$dimensional regular polytopes to combine together, and have created all of the regular polytopes in $\mathbb{R}^4$.

2.1 Another Construction of the Universal Regular Polytopes in $\mathbb{R}^n$

The three regular polytopes that exist in any dimension are, as we shall see later, the simplex, cube, and cross polytope.

The simplex is easily constructed by taking any $n + 1$ points that are equidistant from each other and taking their convex hull. The *convexhull* of an object is the smallest convex set that contains the object-recall the definition of *convex* from section 1. The simplex is self-dual.

The cube is easily constructed by taking the Minkowski sum of the standard n basis vectors. The Minkowski sum is a binary operation on vectors formed by taking one vector and placing the second vector at every point on the first: the shape traced out is the Minkowski sum. For example, the Minkowski sum of two arbitrary vectors is a parallelogram, the Minkowski sum of two orthogonal vectors is a rectangle, and the Minkowski sum of two orthogonal vectors of the same length is a square. The Minkowski sum of three vectors is formed by taking the Minkowski sum of two of them and then tracing that along the third vector; this generalizes naturally to higher dimensions.

The cross polytope, or cocube (so called because it is the dual of the cube) is easily constructed by taking the convex hull of the positive and negative standard n basis vectors. This is where the name cross polytope comes from: the positive and negative basis vectors form a cross, which then defines the polytope.

It is good to see that there are multiple methods for constructing the regular polytopes. It shows us that these really are special, existant things that we can discover, and not just nonsense we're making up. There exist many very visually aesthetic and easy to understand construction of the $4-$dimensional versions of the three regular polytopes that exist in any dimension, for examples, see the bibliography or the presentation accompanying this paper. By direct inspection of these pictures, and the statue of the Octacube, we can actually observe many wonderful facts about these shapes, such as their number of cells, faces, edges, and vertices.

3 A Construction of the Regular Polytopes in $\mathbb{R}^5$...

In order to create the five dimensional regular polytopes we need to know the dihedral angles of the four dimensional regular polyopes. To calculate these, we shall recall our earlier dihedral formula, only this time using the 3 dimensional faces, or *cells*, of our 4 dimensional polytopes.

As the simplex is made up of three tetrahedrons at each edge, its dihedral angle is $\approx 75.5^o$. The hypercube, made up of three cubes at each edge, has dihedral angle of exactly 90^o (this is true in any dimension, see the construction in section 2.1 if you are unconvinced). Both the cocube and octacube, made up of four tetrahedrons and three octahedrons at each edge respectively, have dihedral angle of 120^o. To calculate the cocube's dihedral angle we employ the same trick we used to calculate the dihedral angle of the octahedron: by observing that the vertex figure of the cocube is the octahedron we can truncate it in a similar manner, giving us only three cells meeting at an edge. The 120 cell is made up of larger faces than the octacube and thus has dihedral angle greater than 120^o, while the 600 cell has five tetrahedrons at each edge, more than the cocube, and thus also has dihedral angle greater than 120^o.

Therefore we cannot build anything out of the 120 cell or 600 cell, for the same reason we couldn't make anything out of the heptagon in section 1.2. The octacube and cocube can tile four-space, but can't form new five-dimensional figures; recall the hexagon from section 1.2.

Ahh, but the simplex and tesseract, now those are another story! We can fit three of the tesseract around a face, giving us the new five dimensional cube. Taking the dual of that gives us the cocube. The simplex we can fit three or four of around a face, giving us the new simplex and the cross polytope again. The new simplex is, as always, self-dual, and thus we have all the 5 dimensional regular convex polytopes: the simplex, cube, and cross polytope.

3.1 ... and Higher Dimensions

Recall from 1.3 the fact that if $\{p, q, r, s, t\}$ represents a regular polytope, then $\{q, r, s, t\}\{p, q, r, s\}$ and so on must be regular polytopes. Then by observing the five and four dimensional regular polytopes it becomes apparent that the only polytopes that exist in the higher dimensions are $\{3, 3, ..., 3, 3\}\{4, 3, ..., 3, 3\}$ and $\{3, 3, ..., 3, 4\}$, or the simplex, cube, and cross polytope, respectively. And thus we finish our complete classification of the regular convex polytopes in all of the dimensions.

References

[1] Étienne Ghys, Dimensions: A Walk Through Mathematics! This film gives a good visual introduction to higher dimensions. I recommend it and the novel "Flatland: A Romance In Many Dimensions". Both are available online for free, legally: Flatland is in the public domain and Dimensions is a public-education project.

[2] Paul Kunkel http://whistleralley.com/polyhedra/derivations.htm Last updated August 10, 2004. This informative site contains a derivation of the dihedral formula used here.

[3] *Octacube*, a statue by Prof. Adrian Ocneanu, lies in the entryway of the mathematics building at the Pennsylvania State University. Salvador Dali's masterpiece *Crucifixion* (*Corpus Hypercubus*) hangs in the Metropolitan Museom of Art in New York.

[4] H.S.M. Coxeter, "Regular Polytopes" second edition, Macmillan Mathematics Paperbacks, originally published in 1948. This book is definitive on the subject and is highly recommended, it was the main source used for this paper. Though no specific passages are cited it was invaluable as a background resource.

[5] Euclid's *Elements*, published circa 300 BCE provides the first formal treatment of the regular polytopes. It climaxes with a proof of the uniqueness of the 3 dimensional regular polyhedra-in fact, by coincidence, the same proof presented here.

4 About the Author

4.1 Thomas Eliot

Thomas attended the Mathematics Advanced Study Semester, or MASS Program, at Penn State in Fall 2009. The majority of the research for this paper was done there as his project for his Explorations in Convexity class, as taught by Prof. Sergei Tabachnikov. In Spring 2010 he attended the Budapest Semesters in Mathematics program. He has sadly not been accepted by any of the multitudes of REUs he's applied to, and therefore has had to do all of his research on his own: in the summer of 2009 he invented a voting system, wrote it up, and presented it at MathFest 2009, where it won the ΠME Award for Outstanding Undergraduate Research and Presentation. It is currently being reviewed for publication by the Pi Mu Epsilon journal. In the summer of 2010 he finished this paper and presented it at MathFest 2010, before submitting it to this magazine. His email is `teliot@willamette.edu`. He will graduate in the Spring of 2011 and is eager to discuss his graduate school prospects with anyone willing to do so. He wrote this in the third person because the style guide said to. His mailing address is

Thomas Eliot
8398 E Costilla Ave
Centennial, Co. 80112